NEW SIMILARITY INTEGRALS IN HEAT AND MASS TRANSFER PROCESSES

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Аннотация—В работе получены новые интегралы подобия в задачах теплообмена и массообмена. Эти интегралы выражают собой подобие вихревых, температурных и концентрационных полей. При этом рассматриваются как случай несжимаемого, так и сжимаемого газа, как случай поверхностных, так и объёмных реакций.

NOMENCLATURE

x, co-ordinate along the wall surface;

y, co-ordinate normal to the wall surface;

u, velocity component in x-direction;

v, velocity component in y-direction;

ψ, stream function;

c, concentration;

T, absolute temperature;

t, stagnation temperature $= T + u^2/2c_p$;

p, pressure;

ρ, density;

μ, dynamic viscosity coefficient;

ν, kinematic viscosity coefficient $= \mu/\rho$;

c_p, specific heat at constant pressure;

D, diffusion coefficient;

λ, coefficient of thermal conductivity;

R, gas constant;

Pr, Prandtl number $= c_p\mu/\lambda$;

σ, diffusion number $= \mu/\rho D$;

ω, vorticity $= \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}$;

V, rate of a volumetric chemical reaction;

q, thermal effect of the reaction;

β, relation of activation energy to a gas constant;

$2H$, channel breadth;

A, B, C, K, L, M, a, b, m, n, α, a_1, α_0, α_1, α_2, c_1 and c_2, constants.

Indices

0, corresponds to the surface in a flow.

The quantities at the boundary layer are barred.

1. INTRODUCTION

PATTERNS of viscous-gas or liquid flows accompanied by heterogeneous (surface) or homogeneous (volumetric) chemical reactions are often encountered in practice. And hydrodynamic factors essentially influence the course of chemical reactions since reaction rates depend on concentrations which in their turn depend on the velocity field. It is evident on the same grounds that hydrodynamic factors influence heat-transfer processes. Because of this a detailed consideration of hydrodynamic, thermodynamic and chemical factors becomes essential in a number of applied fields. In view of the great difficulty of integration of the non-linear system of partial differential equations which are to be considered in this paper, it is very important to obtain similarity integrals. A number of new similarity integrals of temperature, concentration and vortex fields both for a compressible and incompressible gas or liquid are obtained in this paper.

2. SIMILARITY INTEGRALS OF VORTEX, CONCENTRATION AND TEMPERATURE FIELDS FOR HETEROGENEOUS REACTIONS

A system of hydrodynamic and diffusion equations may be written as:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

3D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3)$$

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right). \qquad (4)$$

For the time being compressibility is not taken into account in this section and the density ρ may be assumed equal to unity. Generalization for a compressible gas will follow below.

Equations (1) and (2) represent the Navier–Stokes equations for a viscous liquid, equation (3), an incompressibility condition, and equation (4) serves for determining concentrations.

Note that for gases the diffusion number σ is assumed equal to unity, and is sufficiently close to reality.

For the system of equations (1–4) only one integral $c = au + b$ is known which is valid only for a specific case of a boundary layer on a plate and, moreover, connected with the condition of constant concentration and, consequently, with constant chemical reaction rate at the surface.

A new integral is obtained in the present paper and has been found to be applicable to a general case where a pressure gradient is present (flow past a curvilinear profile with a variable concentration at a surface).

Introduce a new dependent variable

$$\chi = c - A\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right). \qquad (5)$$

By virtue of equations (1, 2 and 4) one obtains the equation for χ

$$u\frac{\partial \chi}{\partial x} + v\frac{\partial \chi}{\partial y} = \mu\left(\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2}\right). \qquad (6)$$

The following integral corresponds to this equation

$$\chi = B = \text{const.} \qquad (7)$$

or

$$c = A\omega + B. \qquad (8)$$

Integral (8) represents the similarity integral of eddy and concentration fields for flow past a profile of an arbitrary form.

Let us elucidate to what boundary conditions integral (8) corresponds. For practical applica-

tions it is most convenient to take advantage of a boundary-layer pattern. At the boundary of a layer $(y = \infty)$, $\omega = 0$; $c = \bar{c}$, consequently, $B = \bar{c}$. Then according to equation (1) written in the form of a boundary layer the boundary condition at the surface of a profile for an eddy gradient is

$$\left(\frac{\partial \omega}{\partial y}\right)_0 = \frac{1}{\mu}\frac{dp}{dx}. \qquad (9)$$

In the considered case of a surface reaction we have for the concentration gradient

$$\left(\frac{\partial c}{\partial y}\right)_0 = r(x). \qquad (10)$$

Here $r(x)$ is the given function determining the law of surface solubility.

Comparison of equations (8), (9) and (10) relates $r(x)$ to dp/dx as:

$$r(x) = \frac{A}{\mu}\frac{dp}{dx}. \qquad (11)$$

For example, for a gas flow in a plane channel $(dp/dx = \text{const.})$ we obtain $r(x) = \text{const.}$ and, since $u = \bar{u}(1 - y^2/H^2)$ (where $\bar{u}$ is the axial velocity), then integral (8) gives

$$c = \bar{c} - \frac{2A\bar{u}}{H^2}y.$$

In a similar way the solutions are obtained for a gas flow accompanied by a surface reaction and, in other cases, for a polygon velocity distribution, power velocity distribution, flow in a convergent channel etc.

Let us show that for a specific case of flow past a plate we obtain a more general integral of equation (6). In this case for equation (1) written in the form of a boundary layer it should be assumed that $dp/dx = 0$. If one compares the equations obtained it is possible to get the integral:

$$\chi = Ku + B \qquad (12)$$

or

$$c = A\omega + Ku + B. \qquad (13)$$

At the boundary layer we have

$$u = \bar{u}; \qquad \omega = 0$$

and, consequently,

$$B = \bar{c} - K\bar{u}.$$

Then at the surface we have

$$\left(\frac{\partial \omega}{\partial y}\right)_0 = 0; \qquad \left(\frac{\partial c}{\partial y}\right)_0 = r(x).$$

Thus according to equation (13)

$$r(x) = K\left(\frac{\partial u}{\partial y}\right)_0$$

and since u is determined by the known Blasius formula in the form

$$u = \bar{u} f\left[y\sqrt{\left(\frac{\bar{u}}{\mu x}\right)}\right]$$

where f is the known function, then

$$r(x) = \frac{a}{\sqrt{x}} \tag{14}$$

where

$$a = Kf'(0)\bar{u}\sqrt{\left(\frac{\bar{u}}{\mu}\right)} = 0.332\,K\bar{u}\sqrt{\left(\frac{\bar{u}}{\mu}\right)}.$$

Now we will derive the similarity integrals of vortex and temperature fields. It is known that in the presence of heat conductance at the surface of a profile in a flow the equations for thermodynamics and hydrodynamics of viscous media have an integral of similarity for velocity and temperature fields.

This integral is valid, however, only for a flow past a plate and at the constant temperature of the surface. In the present paper other integrals of thermodynamic equations are obtained and valid both for a flow past a profile of an arbitrary form and for a variable surface temperature.

Equations of motion, incompressibility and heat balance for a viscous gas flow past a profile of an arbitrary form may be written as [1]:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{15}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

$$c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) =$$

$$\frac{\lambda}{\rho}\frac{\partial^2 T}{\partial y^2} + \nu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}u. \tag{16}$$

(At first they are given in the form of boundary layer equations where compressibility is not taken into account; generalization for equations in the Navier–Stokes form as well as that for the case of compressibility is given below.) Introduce the new variable

$$S(x; y) = T + \frac{u^2}{2c_p} - A\omega = t - A\omega \tag{17}$$

where $\omega = \partial u/\partial y$ is a vortex in the boundary layer.

Having differentiated equation (15) with respect to y and having added the equation obtained to equation (15), multiplied by $u/2$, and having made some transformations to equation (16), we obtained the following equation for the function $S(x; y)$ (Pr as usual equals unity).

$$u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} = \nu\frac{\partial^2 S}{\partial y^2}. \tag{18}$$

Since at $y = \infty$ (at the boundary of a layer) ω equals zero and for the stagnation temperature

$$\bar{\imath} = \bar{T} + \frac{\bar{u}^2}{2c_p} = \bar{\imath} = \text{const.}$$

the boundary conditions at infinity for the variable S is written as:

$$S = \bar{\imath} = \text{const.} \quad \text{at } y = \infty. \tag{19}$$

By virtue of equation (15) we get at the surface of a profile (at $y = 0$)

$$\left(\frac{\partial \omega}{\partial y}\right)_0 = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{9}$$

Let the temperature gradient distribution along the surface have the form

$$\left(\frac{\partial T}{\partial y}\right)_0 = r(x) \tag{20}$$

and let

$$r(x) = \frac{A}{\mu} \frac{dp}{dx}. \qquad (11)$$

Then for S the boundary condition at the surface will be:

$$\left(\frac{\partial S}{\partial y}\right)_0 = 0. \qquad (21)$$

The system of equations (18), (19) and (21) may be satisfied by the integral

$$S = T + \frac{u^2}{2c_p} - A\omega = i = \text{const.} \qquad (22)$$

If we introduce a relative stagnation temperature by the formula

$$t_l = T + \frac{u^2}{2c_p} - \bar{T} - \frac{\bar{u}^2}{2c_p}$$

and designate the values corresponding to the profile surface through t_{l0}, ω_0, then integral (22) may be written in the form:

$$\frac{t_l}{t_{l0}} = \frac{\omega}{\omega_0} \qquad (23)$$

i.e. we obtain the similarity integral of temperature and vortex fields.

For a specific case of a flow around a plate ($dp/dx = 0$) we may get a more general integral of equation (17), viz: comparison of equation (17) with equation (15) makes it possible to obtain the integral

$$S = Ku + B. \qquad (22')$$

At the boundary of a layer $u = \bar{u}$; $\omega = 0$; $S = i = \text{const.}$ and, consequently:

$$B = i - K\bar{u}.$$

Thus, integral (22') may be rewritten in the form:

$$S = T + \frac{u^2}{2c_p} - A\omega = i + K(u - \bar{u}). \qquad (22'')$$

For a heat flow through the surface we get:

$$\left(\frac{\partial T}{\partial y}\right)_0 = K\left(\frac{\partial u}{\partial y}\right)_0 \sim \frac{1}{\sqrt{x}}.$$

Consider some specific cases:

(a) *Plate in a flow*. In this case $dp/dx = 0$. By virtue of equations (20) and (11) it corresponds to the condition

$$\left(\frac{\partial T}{\partial y}\right)_0 = 0 \qquad (24)$$

i.e. a heat flow through a surface is absent. According to integral (22) the temperature distribution along the surface has the form:

$$T_0(x) = i + A\omega_0(x).$$

Since for a plate the velocity distribution in a flow is determined by the formula $u = F(y/\sqrt{x})$, then

$$\omega_0(x) = \frac{m}{\sqrt{x}}$$

and finally we get

$$T_0 = i + \frac{C}{\sqrt{x}} \qquad (25)$$

Here $C = Am$ is some constant

$$[m = 0.332 \, \rho\bar{u} \, \sqrt{(\nu\bar{u})}]$$

From equations (24) and (25) it follows that a heat flow through the surface will be absent when the temperature is distributed along the surface according to formula (25). Note that if instead of similarity integral (22), we avail ourselves of generalized integral (22'), then we may directly obtain a solution corresponding to the presence of a heat flow through the surface.

Since the vortex distribution in a flow is expressed by the formula

$$\omega = \frac{1}{\sqrt{x}} F'(y/\sqrt{x})$$

then according to equation (22) we find for the temperature distribution in a flow

$$T = i + \frac{A}{\sqrt{x}} F'(y/\sqrt{x}) - \frac{F^2(y/\sqrt{x})}{2C_p}. \qquad (26)$$

The function $F(y/\sqrt{x})$ entering formula (26) represents the known Blasius function.

(b) *Flow in a converging channel*. In this case for the velocity distribution at the boundary of a layer we have

$$\bar{u}(x) = -\frac{C}{x}$$

and, consequently, for the pressure gradient

$$\frac{dp}{dx} = \frac{\rho C^2}{x^3}.$$

According to equations (20) and (11)

$$\left(\frac{\partial T}{\partial y}\right)_0 = \frac{AC^2}{\nu x^3}. \tag{27}$$

Since for a converging channel the velocity distribution in a flow is determined by the formula

$$u = \frac{1}{x} \varphi \left(\frac{y}{x}\right), \quad \text{where}$$

$$\varphi \left(\frac{y}{x}\right) = - C \left\{ 3th^2 \left[\sqrt{\left(\frac{C}{2\nu}\right)} \frac{y}{x} + 1\cdot146 \right] - 2 \right\} \tag{28}$$

then

$$\omega_0(x) = \frac{1}{x^2} \varphi'(0) = - \frac{2C}{x^2} \sqrt{\left(\frac{C}{3\nu}\right)}.$$

According to equation (22) the temperature distribution along the surface has the form

$$T_0(x) = i - \frac{b}{x^2} \tag{29}$$

where $b = 2CA \sqrt{(C/3\nu)}$ and is some constant.

According to equations (22) and (28) the temperature distribution in a flow is found with the help of the formula

$$T(x; y) = i + \frac{A}{x^2} \varphi'\left(\frac{y}{x}\right) - \frac{\varphi^2(y/x)}{2c_p x^2}.$$

(c) *Polygonal velocity distribution.* For the polygonal velocity distribution at the boundary of a layer we have $\bar{u} = a_1 x$. In this case $dp/dx = - \rho a_1^2 x$. By virtue of equations (20) and (11) we get for the temperature gradient at the surface

$$\left(\frac{\partial T}{\partial y}\right)_0 = - \frac{A}{\nu} a_1^2 x. \tag{30}$$

Since in the case considered the velocity distribution in a flow is determined by the formula $u = a_1 x f'[\sqrt{(a_1/\nu)}y]$ where f is the known function, then

$$\omega_0(x) = a_1 \sqrt{(a_1/\nu)}x f''(0) = 1\cdot23 \, a_1 \sqrt{(a_1/\nu)}x.$$

According to equation (22) the temperature distribution along the surface has the form:

$$T_0(x) = i + Bx \tag{31}$$

where $B = 1\cdot23 a_1 \sqrt{(a_1/\nu)}A$ and is some constant.

According to equation (22) the temperature distribution in a flow is of the form:

$$T(x; y) = i + Aa_1 \sqrt{(a_1/\nu)}xf''[\sqrt{(a_1/\nu)}y]$$
$$- a_1^2 x^2 \frac{f'^2[\sqrt{(a_1/\nu)}y]}{2c_p}. \tag{32}$$

(d) *Power velocity distribution.* If at the boundary of a layer there is a power velocity distribution, $\bar{u} = ax^m$, then for the pressure gradient we have [2]

$$\frac{dp}{dx} = - \rho a^2 m x^{2m-1}.$$

By virtue of equations (20) and (11) we find the temperature gradient distribution along the surface in the form

$$\left(\frac{\partial T}{\partial y}\right)_0 = - \frac{A}{\nu} a^2 m x^{2m-1}. \tag{33}$$

The velocity distribution in a flow is determined by the formula

$$u = \sqrt{\left(\frac{2}{m+1}\right)} \bar{u} q' \left[\sqrt{\left(\frac{\bar{u}}{\nu x}\right)} y \right]$$

where q is the known function.

For the vortex distribution over the surface we find

$$\omega_0(x) = \frac{2}{m+1} \bar{u} \sqrt{\left(\frac{\bar{u}}{\nu x}\right)} q''(0).$$

According to equation (22) the temperature distribution along the surface has the form

$$T_0(x) = i + Bx^{(3m-1)/2} \tag{34}$$

where $B = aA \sqrt{[2a/(m+1)\nu]} q''(0)$ and is some constant.

The temperature distribution in a flow is as follows:

$$T(x; y) = i + A \sqrt{\left(\frac{2}{m+1}\right)} \bar{u} \sqrt{\left(\frac{\bar{u}}{\nu x}\right)}$$

$$q'' \left[\sqrt{\left(\frac{\bar{u}}{\nu x}\right)} y \right] - \frac{\bar{u}^2 q'^2[\sqrt{(\bar{u}/\nu x)}y]}{(m+1)c_p}. \tag{35}$$

For $m = \frac{1}{2}$ the heat flow through the surface is constant; for $m = \frac{1}{3}$ the temperature along the surface is constant; for $m = -0\cdot0904$ the temperature along the surface is constant as well since calculations show that in this case $q''(0) = 0$.

Let us generalize the results given for an incompressible viscous liquid flow at arbitrary Reynolds numbers (i.e. for the equation of motion in the Navier–Stokes form).

Equations for the temperature and vortices may be written in the form [3]:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\lambda}{c\rho} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \qquad (36)$$

$$u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \mu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \qquad (37)$$

for an incompressible liquid or gas the dissipation term is omitted.

For $Pr = 1$ we get the integral

$$T = A\omega + \bar{T}. \qquad (38)$$

This integral takes place when the corresponding boundary conditions for the temperature are fulfilled.

Consider some specific cases:

(e) *Liquid flow between two parallel walls.* As is known, in this case

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \text{const.} \quad u = -\frac{1}{2\mu} \frac{\mathrm{d}p}{\mathrm{d}x} (H^2 - y^2)$$

$$v = 0 \qquad \omega = \frac{1}{\mu} \frac{\mathrm{d}p}{\mathrm{d}x} y.$$

According to equation (38) we get

$$T = \frac{A}{\mu} \frac{\mathrm{d}p}{\mathrm{d}x} y + \bar{T}. \qquad (39)$$

The solution corresponds to the constant temperature and constant temperature gradient along the surface.

(f) *Couette flow.* In this case $u = my/H$; $v = 0$; $\omega = m/H$. Due to equation (38) the solution corresponds to constant temperature according to the formula

$$T = \bar{T} + A(m/H). \qquad (40)$$

In a similar way the temperature distribution may be found for a viscous liquid flow at arbitrary Reynolds numbers in a converging channel and for other specific cases.

3. SIMILARITY INTEGRALS OF FRICTION, TEMPERATURE AND CONCENTRATION FIELDS IN A BOUNDARY LAYER OF A COMPRESSIBLE GAS

If we consider a boundary layer of a compressible gas on a plate, then for this case in the presence of the usual condition $Pr = (c\rho\mu/\lambda) = 1$ two similarity integrals for the stagnation temperature are known, viz.

(i) $$t = \frac{u^2}{2c_p} + T = b$$

(ii) $$t = \frac{u^2}{2c_p} + T = au + b.$$

However, since at the surface in the flow there is a boundary condition $u = 0$, then both integrals give $T_0 = b$, i.e. they correspond to a constant temperature of the surface. The present paper gives new similarity integrals corresponding indeed to the variable temperature of the surface (similarity integral of friction and temperature fields) and to the variable concentration along the surface (similarity integral of friction and concentration fields), both the former and latter integrals being obtained when the gas compressibility is taken into account (earlier compressibility was not taken into account).

A system of hydrodynamic and thermodynamic equations for the boundary layer of a compressible gas on a plate ($Pr = 1$) may be written as:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \qquad (41)$$

$$\rho u = \frac{\partial \psi}{\partial y}; \qquad \rho v = -\frac{\partial \psi}{\partial x} \qquad (42)$$

$$\rho u \frac{\partial t}{\partial x} + \rho v \frac{\partial t}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial t}{\partial y} \right). \qquad (43)$$

Let us apply the Dorodnitsyn transformation method [4] to the system of equations (41–43), i.e. let us turn from the independent variables x and y to the independent variables

$$x \text{ and } \eta = \int_0^y \rho(x; y) \, \mathrm{d}y. \tag{44}$$

In new variables we get the system of equations

$$\frac{\partial \psi}{\partial \eta} \frac{\partial^2 \psi}{\partial x \partial \eta} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial \eta^2} = \frac{\partial}{\partial \eta} \left(\mu \rho \frac{\partial^2 \psi}{\partial \eta^2} \right) \tag{45}$$

$$\frac{\partial \psi}{\partial \eta} \frac{\partial t}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial t}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\mu \rho \frac{\partial t}{\partial \eta} \right). \tag{46}$$

Assuming, as usual, the linear law of the dependence of viscosity on temperature $\mu = aT$ and using the equation of gas state $\rho = p/RT$ we have

$$\mu \rho = \frac{ap}{R} = \text{const.}$$

Turning then in equations (45) and (46) to a dependent variable $\psi_1 = \psi/\mu\rho$ we reduce them to the form

$$\frac{\partial \psi_1}{\partial \eta} \frac{\partial^2 \psi_1}{\partial x \partial \eta} - \frac{\partial \psi_1}{\partial x} \frac{\partial^2 \psi_1}{\partial \eta^2} = \frac{\partial^3 \psi_1}{\partial \eta^3} \tag{47}$$

$$\frac{\partial \psi_1}{\partial \eta} \frac{\partial t}{\partial x} - \frac{\partial \psi_1}{\partial x} \frac{\partial t}{\partial \eta} = \frac{\partial^2 t}{\partial \eta^2}. \tag{48}$$

By differentiating equation (47) with respect to η and introducing the dependent variable

$$\tau = (\mu \rho)^2 \frac{\partial^2 \psi_1}{\partial \eta^2} \tag{49}$$

we get the equation

$$\frac{\partial \psi_1}{\partial \eta} \frac{\partial \tau}{\partial x} - \frac{\partial \psi_1}{\partial x} \frac{\partial \tau}{\partial \eta} = \frac{\partial^2 \tau}{\partial \eta^2}. \tag{50}$$

Let us clarify the physical meaning of the variable τ. Since

$$u = \frac{1}{\rho} \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}$$

then for τ we have

$$\tau = (\mu \rho)^2 \frac{\partial^2 \psi_1}{\partial \eta^2} = \mu \rho \frac{\partial^2 \psi}{\partial \eta^2} = \mu \rho \frac{\partial u}{\partial \eta} = \mu \frac{\partial u}{\partial y}.$$

Consequently, τ represents the friction stress.

Comparing equations (50) and (48) we get the first integral from the new similarity integrals

$$\tau = At + B. \tag{51}$$

This integral represents the condition of similarity of the friction stress and friction temperature fields in the boundary-layer of a compressible gas.

Since at the boundary of a layer $\tau = 0$ and $t = \bar{t} = \text{const.}$, then $B = -A\bar{t}$, i.e. integral (51) may be rewritten as:

$$\frac{\tau}{t - \bar{t}} = A = \text{const.} \tag{51'}$$

Note that from equation (41) it follows that $(\partial \tau/\partial y)_0 = 0$. Consequently, equation (51) gives

$$(\partial t/\partial y)_0 = (\partial T/\partial y)_0 = 0.$$

i.e. integral (51) corresponds to the absence of a heat flow through the surface of a profile in a flow.

On the basis of equation (51') the temperature distribution in a flow is determined by the formula

$$T = \frac{\tau}{A} + \bar{t} - \frac{u^2}{2c_p}$$

$$= (\mu \rho)^2 \left[\frac{1}{A} \frac{\partial^2 \psi_1}{\partial \eta^2} - \frac{1}{2c_p} \left(\frac{\partial \psi_1}{\partial \eta} \right)^2 \right] + \bar{t}. \tag{52}$$

Here $\psi_1(x; \eta) = \sqrt{(x)}F(\xi)$, where $\xi = \eta/\sqrt{x}$. $F(\xi)$ is determined by the Blasius formula.

Introducing the variable ξ into formula (52) we have

$$T = (\mu \rho)^2 \left[\frac{1}{A\sqrt{x}} F''(\xi) - \frac{1}{2c_p} F'^2(\xi) \right] + \bar{t}. \tag{52'}$$

Since at the surface $F'(0) = 0$, we get for the temperature of a surface

$$T_0 = \bar{t} + \frac{m}{\sqrt{x}}. \tag{53}$$

Consequently, similarity integral (51) corresponds to the variable surface temperature.

Comparing equations (50), (47) and (48) we obtain another similarity integral

$$\tau = At + B + C \frac{\partial \psi_1}{\partial \eta}. \tag{54}$$

Since at the boundary of a layer $\tau = 0$; $t = \bar{t} = \text{const.}$, $\partial \psi_1/\partial \eta = \bar{u}/\mu\rho = \text{const.}$, then

$B = -A\bar{t} - C(\bar{u}/\mu\rho)$ and integral (54) may be rewritten as

$$\tau = A(t - \bar{t}) + \frac{C}{\mu\rho}(u - \bar{u}). \qquad (55)$$

Integral (55) shows that the friction stress is expressed by the linear combination of the stagnation temperature and velocity drops.

Owing to the fact that at the surface $(\partial\tau/\partial y)_0 = 0$ and since $(\partial t/\partial y)_0 = (\partial T/\partial y)_0$ we get

$$\left(\frac{\partial T}{\partial y}\right)_0 = -\frac{C}{\mu\rho A}\left(\frac{\partial u}{\partial y}\right)_0 \qquad (56)$$

i.e. integral (55) corresponds to the presence of a heat flow through the surface of a profile in a flow.

On the basis of equation (55) the temperature distribution in a flow is defined by the formula

$$T = \frac{\tau}{A} + \bar{t} - \frac{u^2}{2c_p} - \frac{C}{\mu\rho}(u - \bar{u})$$

$$= (\mu\rho)^2\left[\frac{1}{A}\frac{\partial^2\psi_1}{\partial\eta^2} - \frac{1}{2c_p}\left(\frac{\partial\psi_1}{\partial\eta}\right)^2\right]$$

$$- C\frac{\partial\psi_1}{\partial\eta} + \bar{t} + \frac{C}{\mu\rho}\bar{u}. \qquad (57)$$

Substituting $\psi_1(x;\eta)$ for $\sqrt{(x)}F(\xi)$ we obtain

$$T = (\mu\rho)^2\left[\frac{F''(\xi)}{A\sqrt{x}} - \frac{1}{2c_p}F'^2(\xi)\right]$$

$$- CF'(\xi) + \bar{t} + \frac{C}{\mu\rho}u \qquad (57')$$

and since at the surface $F'(0) = 0$ we finally get

$$T_0 = \bar{t} + \frac{m}{\sqrt{x}} \qquad (58)$$

that again corresponds to the variable temperature distribution along the surface.

Substituting in formula (56) $(\partial u/\partial y)_0$ for $(\mu\rho/\sqrt{x})\rho_0 F''(0)$, we have

$$\left(\frac{\partial T}{\partial y}\right)_0 = \frac{K}{\sqrt{x}}. \qquad (59)$$

Formula (59) determines the temperature gradient distribution along the surface.

Now obtain analogous integrals for similarity of concentration and friction fields.

An equation for concentration $c(x;y)$ may be written in the form

$$\rho u\frac{\partial c}{\partial x} + \rho v\frac{\partial c}{\partial y} = \frac{\partial}{\partial y}\left(\rho D\frac{\partial c}{\partial y}\right). \qquad (60)$$

Turning from the independent variables x and y to the independent variables

$$x \text{ and } \eta = \int_0^y \rho(x;y)\,dy \qquad (44)$$

we get the equation in new variables

$$\frac{\partial\psi}{\partial\eta}\frac{\partial c}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial c}{\partial\eta} = \frac{\partial}{\partial\eta}\left(\rho^2 D\frac{\partial c}{\partial\eta}\right). \qquad (61)$$

Assuming that the diffusion number $\mu/\rho D = 1$ and that $\mu\rho = \text{const.}$, we may reduce the equation to the form

$$\frac{\partial\psi_1}{\partial\eta}\frac{\partial c}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial c}{\partial\eta} = \frac{\partial^2 c}{\partial\eta^2}. \qquad (62)$$

Comparison of equations (62) and (50) gives the similarity integral

$$\tau = Kc + L, \qquad (63)$$

which expresses the condition of similarity of the friction stress and concentration fields.

Since at the boundary of a layer $\tau = 0$ and $c = \bar{c} = \text{const.}$, then $L = -K\bar{c}$ and integral (63) may be otherwise written as

$$\frac{\tau}{c - \bar{c}} = K = \text{const.} \qquad (64)$$

It follows from equation (41) that $(\partial\tau/\partial y)_0 = 0$, whence we get $(\partial c/\partial y)_0 = 0$, i.e. integral (63) corresponds to the absence of a heterogeneous chemical reaction on the surface of a profile. However, if we use the generalized similarity integral which will be obtained, below then it will correspond to the presence of a heterogeneous reaction at the surface of a profile. To obtain this integral we compare equations (62), (50) and (47) and we arrive at the similarity integral

$$\tau = Kc + L + M\frac{\partial\psi_1}{\partial\eta}. \qquad (65)$$

The condition at the boundary of a layer gives

$$c = \bar{c}; \quad \tau = 0; \quad \frac{\partial\psi_1}{\partial\eta} = \frac{\bar{u}}{\mu\rho} = \text{const.}$$

and integral (65) may be written as

$$\tau = K(c - \bar{c}) + \frac{M}{\mu\rho}(u - \bar{u}). \qquad (66)$$

The condition at the surface gives $(\partial\tau/\partial y)_0 = 0$. Consequently,

$$\left(\frac{\partial c}{\partial y}\right)_0 = -\frac{M}{K\mu\rho}\left(\frac{\partial u}{\partial y}\right)_0 \qquad (67)$$

i.e. the concentration gradient at the surface differs from 0, and this corresponds to the presence of a heterogeneous reaction.

4. SIMILARITY INTEGRALS FOR HOMOGENEOUS REACTIONS

Consider a system of differential equations corresponding to a plane steady flow round a curvilinear surface. Moreover, chemical reactions occur both in the volume of the viscous gas (liquid) and at the surface. For convenience of consideration we use a usual pattern of a boundary layer (note that a number of the results obtained may be easily generalized in the Navier–Stokes form for equations of motion of a viscous liquid). Thus we have the following system of partial differential equations [5, 6]:

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = \frac{\partial}{\partial y}\left(D\frac{\partial c}{\partial y}\right) + V(c) \qquad (68)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad (69)$$

$$u = \frac{\partial\psi}{\partial y}; \qquad v = -\frac{\partial\psi}{\partial x} \qquad (70)$$

μ and D are usually assumed to be constants; below, however, a case will be considered when these coefficients are variable; $V(c)$ is the volumetric rate of chemical reaction which, in case of an isothermal flow, is a function of concentration alone and in the case of a non-isothermal flow depends both on concentration and temperature (the temperature effect on the chemical reaction rate is determined by the Arrhenius formula and in future will be taken into account); the density ρ for an incompressible gas may be assumed equal to unity.

The boundary conditions for the system of equations (69, 70) are well-known. As to the boundary conditions for equation (68) they depend on the specific chemical conditions of the problem and will be considered further.

In equation (68) let us turn to independent variables x, $u(x; y)$ and to the dependent variable $E = \mu(\partial u/\partial y)^2$ (here E is the energy loss in a boundary layer). After the calculation of derivatives using equation (69) we come to the equation:

$$E\frac{\partial^2 c}{\partial u^2} + \frac{1 - \sigma}{2}\frac{\partial E}{\partial u}\frac{\partial c}{\partial u}$$

$$+ \sigma\left[\frac{dp}{dx}\frac{\partial c}{\partial u} - u\frac{\partial c}{\partial x} + V(c)\right] = 0. \quad (71)$$

In case $\sigma = 1$ equation (71) acquires a simpler form

$$E\frac{\partial^2 c}{\partial u^2} + \frac{dp}{dx}\frac{\partial c}{\partial u} - u\frac{\partial c}{\partial x} + V(c) = 0. \qquad (72)$$

Now transform the system of equations (69, 70) to new variables. After calculating the derivatives in equation (69) and using equation (70) we arrive at the equation

$$E\frac{\partial^2\tau}{\partial u^2} + \frac{dp}{dx}\frac{\partial\tau}{\partial u} - u\frac{\partial\tau}{\partial x} = 0. \qquad (73)$$

Here $\tau = \mu(\partial u/\partial y) = \sqrt{(\mu E)}$ is the friction stress. Equation (73) may be rewritten as:

$$E\frac{\partial^2 E}{\partial u^2} - \frac{1}{2}\left(\frac{\partial E}{\partial u}\right)^2 + \frac{dp}{dx}\frac{\partial E}{\partial u} - u\frac{\partial E}{\partial x} = 0. \quad (74)$$

Thus the solution of the problems on hydrodynamics of a viscous gas in the presence of homogeneous reactions is reduced to that of the system of equations (71) or (72) and (73) or (74) at the corresponding boundary conditions.

Consider the integration of the system of equations (71) and (74), that is consider the most general problem of a flow around an arbitrary profile for an arbitrary law of a volumetric chemical reaction and for an arbitrary number σ. To integrate this system some boundary conditions should be given. For example, if we assume that the surface is insoluble, that is, that a heterogeneous reaction does not take place, then

$$\left(\frac{\partial c}{\partial u}\right)_0 = \frac{(\partial c/\partial y)_0}{(\partial u/\partial y)_0} = 0. \qquad (75)$$

It follows from equation (69) a boundary condition for hydrodynamic equation (74) at the surface may be written in the form

$$\left(\frac{\partial E}{\partial u}\right)_0 = 2\frac{dp}{dx}. \qquad (76)$$

From equation (71) and from the subsequent equations obtained by differentiating equation (71) we determine the following values of the derivatives at the surface in a flow (equation (74) is used as well):

$$\left(\frac{\partial^2 c}{\partial u^2}\right)_0 = -\frac{\sigma V(c_0)}{E_0} \qquad (77)$$

$$\left(\frac{\partial^3 c}{\partial u^3}\right)_0 = \sigma \frac{\dfrac{dc_0}{dx} + 3\dfrac{dp}{dx}\dfrac{V(c_0)}{E_0}}{E_0} \qquad (78)$$

and so on.

The desired solution $c(x; y)$ may be given in the form of a series in powers of u, the coefficients of which are expressed through the obtained values of the derivatives. As to $c_0(x)$, then, it is determined from the boundary condition

$$c(x; \bar{u}) = \bar{c}. \qquad (79)$$

Now determine at what conditions equation (72) will have the similarity integral of the velocity and concentration fields. Let us write this integral in the form

$$c = a(x) + \frac{u}{b(x)} \qquad (80)$$

where in accordance with the boundary conditions

$$a(x) = c_0(x); \quad b(x) = \frac{\bar{u}(x)}{\bar{c} - c_0(x)}.$$

Substituting equation (80) into equation (72) we get

$$V(c) = -\left[\frac{1}{b(x)}\frac{dp}{dx} + b(x)a(x)\frac{da}{dx} + \frac{db}{dx}a^2(x)\right]$$
$$+ \left[b(x)\frac{da}{dx} + 2a(x)\frac{db}{dx}\right]c - \frac{db}{dx}c^2. \qquad (81)$$

Since the coefficients of the powers of c on the right-hand side should be constant, then the following system of equations is obtained

$$\frac{db}{dx} = -a_2; \quad b(x)\frac{da}{dx} + 2a(x)\frac{db}{dx} = a_1;$$

$$\frac{dp}{dx} = -b(x)\left[a_0 + b(x)a(x)\frac{da}{dx} + \frac{db}{dx}a^2(x)\right]. \qquad (82)$$

The first of these equations gives

$$b(x) = -a_2 x + c_1$$

the second,

$$a(x) = \frac{c_2}{(c_1 - a_2 x)^2} - \frac{a_1}{2a_2} \text{ (for } a_2 \neq 0)$$

$$a(x) = \frac{a_1}{c_1}x + c_2 \qquad \text{(for } a_2 = 0).$$

The pressure gradient dp/dx and, consequently, the distribution $\bar{u}(x)$ is determined from the third equation.

Note that the reaction considered for the rate of a volumetric chemical reaction

$$V(c) = a_0 + a_1 c + a_2 c^2 \qquad (83)$$

is sufficiently general since it generalizes the most important and widespread practical cases (e.g. for the reaction of the first order $a_0 = a_2 = 0$, for the reaction of the second order $a_0 = a_1 = 0$).

Now consider the same problem allowing for the effect of temperature on the chemical reaction rate. Then, in addition to hydrodynamic and diffusion equations we should consider also a heat balance equation taking into account the heat released during the chemical reaction. Thus, we should consider the following system of equations (for a reaction of the first order)

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2} + a\exp\left(-\frac{\beta}{T}\right)c \qquad (84)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\lambda}{c_p}\frac{\partial^2 T}{\partial y^2} + qa\exp\left(-\frac{\beta}{T}\right)c \qquad (85)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \mu\frac{\partial^2 u}{\partial y^2} - \frac{dp}{dx} \qquad (69')$$

$$u = \frac{\partial \psi}{\partial y}; \qquad v = -\frac{\partial \psi}{\partial x}. \qquad (70)$$

If the number $c_p D/\lambda$ equals unity, then equations (84) and (85) have the integral

$$T = qc + n. \qquad (86)$$

This integral represents the similarity of the temperature and concentration fields.

Substituting equation (86) into equation (84) we get the equation $(D = \mu)$

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \mu \frac{\partial^2 c}{\partial y^2} + \alpha \exp\left(-\frac{\beta}{qc + n}\right)c. \qquad (87)$$

Then expanding $\exp[-\beta/(\lambda c + n)]$ in powers of c, we may write equation (87) in the form

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \mu \frac{\partial^2 c}{\partial y^2}$$
$$+ \alpha \exp\left(-\frac{\beta}{n}\right)\left(1 + \frac{\beta q}{n^2}c + \ldots\right)c. \qquad (87')$$

When c is small, we may neglect the terms in brackets beginning from c^2 and, consequently, come to the equation

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \mu \frac{\partial^2 c}{\partial y^2} + a_1 c + a_2 c^2 \qquad (88)$$

where

$$a_1 = \alpha \exp\left(-\frac{\beta}{n}\right); \qquad a_2 = a_1 \frac{\beta q}{n^2}.$$

If we seek for the solution of equation (88) in the form of the similarity integral of the concentration and velocity fields:

$$c = c_0(x) + \frac{u}{\bar{u}}[\bar{c} - c_0(x)] \qquad (80')$$

then by virtue of equation (69') we shall obtain the system of equations (82) (where it should be assumed that $\alpha_0 = 0$), from which it follows that integral (80') corresponds to the velocity distribution of a potential flow

$$\bar{u} = -\frac{K}{2}(x + c_1) + \frac{c_2}{x + c_1};$$

where $\qquad K = a_1 + 2a_2 \bar{c}$

and to the distribution of concentrations along the surface

$$c_0(x) = \frac{c_2}{a_2(x + c_1)^2} - \frac{n^2}{2\beta q}.$$

Now we shall take into account the effect of concentration on viscosity and diffusion coefficients. First we shall confine ourselves to the case of a surface reaction and consider a flow over a plate; we shall write a system similar to the system of equations (68–70)

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{\partial}{\partial y}\left[D(c) \frac{\partial c}{\partial y}\right] \qquad (89)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\mu(c) \frac{\partial u}{\partial y}\right] \qquad (90)$$

$$u = \frac{\partial \psi}{\partial y}; \qquad v = -\frac{\partial \psi}{\partial x}. \qquad (70)$$

In equation (90) going over to the independent variables x and $u(x; y)$ and to the dependent variable $\tau = \mu(c)(\partial u/\partial y)$, we obtain the equation

$$\frac{\partial^2 \tau}{\partial u^2} = -u \frac{\partial}{\partial x}\left(\frac{\mu}{\tau}\right). \qquad (91)$$

If we take the condition $\sigma = 1$, at which the system of equations (89, 90) has the integral $c = au + b$ and the power law of the dependence of viscosity on concentration $\mu(c) = Ac^n$, we come to the equation

$$\tau^2 \frac{\partial^2 \tau}{\partial u^2} = Au(au + b)^n \frac{\partial \tau}{\partial x}. \qquad (92)$$

The boundary conditions of equation (92) are

$$\left(\frac{\partial \tau}{\partial u}\right)_0 = 0 \qquad (93)$$

$$\tau(x; \bar{u}) = 0. \qquad (94)$$

The solution of the system of equations (92–94) may be sought in the form:

$$\tau = \sqrt{\left(\frac{A\bar{u}^3 c_0^n}{2x}\right)} \tau_1(u_1) \qquad (95)$$

where $u_1 = u/\bar{u}$ is the dimensionless velocity.

Then we come to an ordinary differential equation

$$\tau_1 \frac{d^2 \tau_1}{du_1^2} = -u_1(1 + mu_1)^n \qquad (96)$$

where $m = (\bar{c}/c_0) - 1$; at boundary conditions

$$\frac{d\tau_1}{du_1} = 0 \qquad (97)$$

$$\tau_1(1) = 0. \qquad (98)$$

The system of equations (96–98) is easily integrated with the help of the series.

In an analogous way in equation (91) we may use our similarity integrals (8) or (13) of concentration and vortex fields. The solution will correspond to the variable concentration along the surface.

Note that the condition $\sigma = 1$ is not an essential restriction since analogous transformations may be carried out too for $\sigma \neq 1$ and when the volumetric reaction takes place. In this case applying transformation of the concentration equation to our variables we derive the following equation

$$\frac{\tau^2}{\mu}\frac{\partial^2 c}{\partial u^2} + (1 - \sigma)\frac{\tau}{\mu}\frac{\partial \tau}{\partial u}\frac{\partial c}{\partial u}$$

$$- \sigma u\frac{\partial c}{\partial x} + \sigma V(c) = 0. \qquad (99)$$

If $V(c) = 0$, i.e. if the volumetric reaction does not take place, then we may assume that $\tau = (1/\sqrt{x})\tau_1(u)$ and $c = c(u)$ and the equation

will be reduced to an ordinary differential equation

$$\tau_1\frac{a^2 c}{du^2} + \frac{1 - \sigma}{\sigma}\frac{d\tau_1}{du}\frac{dc}{du} = 0 \qquad (100)$$

which is integrated in quadratures in the form (α and β are integration constants)

$$c = \alpha \int \tau_1^{(\sigma-1)/\sigma}\,du + \beta. \qquad (101)$$

REFERENCES

1. H. SCHLICHTING, *Boundary Layer Theory*. McGraw-Hill, New York (1960).
2. D. MEKSYN, *New Methods in Laminar Boundary Layer Theory*. Pergamon Press, Oxford–London (1961).
3. A. V. LUIKOV and Y. A. MIKHAILOV, *Theory of Energy and Mass Transfer* (Teoriya perenosa energii i veschchestva). Izd. Akad. Nauk, B.S.S.R., Minsk (1959).
4. A. A. DORODNITSYN, Boundary layer in a compressible gas (Pogranichnyi sloi v szhimaemom gaze). *Applied Math. Mech.* **6**, 449–485 (1942).
5. A. M. FAINZIL'BER, Similarity integrals in the hydrodynamics of heterogeneous and homogeneous processes (Integraly podobiya v gidrodinamike geterogennykh i gomogennykh protsessov). *Dokl. Akad. Nauk S.S.S.R.* **112**, 4, 607–610 (1957).
6. A. M. FAINZIL'BER, An energy form of equations of viscous liquid motion and an integral of slightly vortical flow (Energeticheskaya forma uravnenii dvizheniya vyazkoi zhidkosti i integral malozavikhrennykh techenii). *Dokl. Akad. Nauk S.S.S.R.*, **106**, 5, 793–796 (1956).

Abstract—In the present work new similarity integrals are obtained in heat and mass transfer problems. They represent the similarity of vortex, temperature and concentration fields. Both the case of an incompressible and compressible gas and that of surface and volumetric reactions are considered.

Résumé—Ce travail présente de nouvelles intégrales semblables pour les problèmes de transport de chaleur et de masse. Elles traduisent la similitude des champs de tourbillons, de températures et de concentrations. Le cas d'un gaz compressible ou incompressible et celui des réactions superficielles et volumiques.

Zusammenfassung—Für Wärme- und Stoffübergangsprobleme liessen sich neue Ähnlichkeitsintegrale finden. Sie geben die Ähnlichkeit von Wirbel-, Temperatur- und Konzentrationsfeldern. Verschiedene Möglichkeiten wurden berücksichtigt, nämlich: inkompressibles und kompressibles Gas, Oberflächen- und Raumreaktionen.